

Quasi-Frobenius-splitting and lifting of Calabi-Yau varieties in characteristic p

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Abstract

Extending the notion of Frobenius-splitting, we prove that every finite height Calabi-Yau variety defined over an algebraically closed field of positive characteristic can be lifted to the ring of Witt vectors of length two.

1 Introduction

Let k be an algebraically closed field of characteristic $p > 0$. A smooth proper variety X over k of dimension n is said to be *Calabi-Yau* if $\omega_X \simeq \mathcal{O}_X$ and $H^i(X, \mathcal{O}_X) = 0$ for $0 < i < n$. There are some Calabi-Yau threefolds in positive characteristic which are known to have no liftings to characteristic zero ([H], [Schr], [HIS07], [HIS08], [Scho], [CS]). This phenomena is in contrast to the two dimensional case, i.e., it is well known that every $K3$ surface over k admits a lifting to characteristic zero ([RS], [D]).

Recall that Calabi-Yau varieties over k are classified by the Artin-Mazur height ht which takes a value in positive integers or infinity. For definition and its properties, see §2.3.

For all Calabi-Yau varieties which are known to be non liftable, the Artin-Mazur height is infinity. So one can ask the following:

Question 1.1. *If the Artin-Mazur height of a Calabi-Yau variety over k is finite, then does it admit a lifting to characteristic zero?*

Our main result gives a partial answer to this question:

Theorem 1.2. *Let X be a Calabi-Yau variety over k . If the height of X is finite, then X admits a flat lifting to $W_2(k)$. Here $W_2(k)$ denotes the ring of Witt vectors of length two of k .*

By the work of Deligne-Illusie [DI], we have;

Corollary 1.3. *If $\dim(X) \leq p$, then the Hodge to de Rham spectral sequence degenerates at E_1 .*

Remark 1.4. In [E], Ekedahl announced that Hirokado and Schröer's non-liftable Calabi-Yau varieties admit no liftings to $W_2(k)$.

This result is known for height one Calabi-Yau varieties. Recall that a variety X over k is *Frobenius-splitting* if the absolute Frobenius map

$$F: \mathcal{O}_X \rightarrow F_*\mathcal{O}_X$$

splits as \mathcal{O}_X -modules. This was introduced in [MR] and has applications to the representation theory. Furthermore, the same notion is actively studied in the theory of singularities (it is also called *F-pure* in such an area). For a Calabi-Yau variety X , the height of X is one if and only if X is Frobenius-splitting. In [J], it is proved that any Frobenius-splitting variety admits a flat lifting to $W_2(k)$.

A main ingredient of this paper is to introduce the notion of *quasi-Frobenius-splitting* variety. More precisely, we first define a new invariant $sht(X)$, the *Frobenius-splitting height* of a variety X , which quantifies the notion of Frobenius-splitting. Then we define that X is quasi-Frobenius-splitting when $sht(X)$ is finite. We will show that any smooth quasi-Frobenius-splitting variety admits a lifting to $W_2(k)$ and, for Calabi-Yau varieties, the Artin-Mazur height is equal to the Frobenius-splitting height. For the first statement, the proof is done by the same line of [J]. The second assertion is based on the study of the Artin-Mazur height of Calabi-Yau varieties by G. van der Geer and T. Katsura ([GK]).

2 Preliminaries

Let X be a smooth variety over k and F be the absolute Frobenius of X . $\Omega_X = \Omega_{X/k}^1$ denotes the sheaf of Kähler differentials on X .

2.1 Differential calculus in characteristic p

(The degree 0 and 1 part of) the Cartier isomorphism C is expressed as the following exact sequences of locally free \mathcal{O}_X -modules:

$$\begin{aligned} 0 \rightarrow \mathcal{O}_X &\xrightarrow{F} F_*\mathcal{O}_X \xrightarrow{d} B_1\Omega_X \rightarrow 0, \\ 0 \rightarrow B_1\Omega_X &\rightarrow Z_1\Omega_X \xrightarrow{C} \Omega_X \rightarrow 0. \end{aligned}$$

Here $B_1\Omega_X$ (resp. $Z_1\Omega_X$) is the sheaf of exact (resp. closed) one forms on X . As $d(f^p g) = f^p dg$, we may regard $B_1\Omega_X$ and $Z_1\Omega_X$ as \mathcal{O}_X -submodules of $F_*\Omega_X$.

Following Illusie [I], for $m \geq 1$, we define abelian subsheaves $B_m\Omega_X, Z_m\Omega_X$ inductively by $B_{m+1}\Omega_X := C^{-1}(B_m\Omega_X)$, $Z_{m+1}\Omega_X := C^{-1}(Z_m\Omega_X)$. We regard $B_m\Omega_X$ and $Z_m\Omega_X$ as \mathcal{O}_X -modules so that the inclusions $B_m\Omega_X \subset F_*^m\Omega_X$ and $Z_m\Omega_X \subset F_*^m\Omega_X$ are morphisms of \mathcal{O}_X -modules.

Let $W_m\mathcal{O}_X$ be the sheaf of Witt vectors of length m . This is a sheaf of rings on X and a local section of $W_m\mathcal{O}_X$ is expressed as a m -tuple of regular

functions on X . Note that $W_1\mathcal{O}_X$ is \mathcal{O}_X . We have three operators

$$\begin{aligned} F: W_m\mathcal{O}_X &\rightarrow W_m\mathcal{O}_X; (f_0, \dots, f_{m-1}) \mapsto (f_0^p, \dots, f_{m-1}^p), \\ V: W_m\mathcal{O}_X &\rightarrow W_{m+1}\mathcal{O}_X; (f_0, \dots, f_{m-1}) \mapsto (0, f_0, \dots, f_{m-1}), \\ R: W_{m+1}\mathcal{O}_X &\rightarrow W_m\mathcal{O}_X; (f_0, \dots, f_m) \mapsto (f_0, \dots, f_{m-1}) \end{aligned}$$

and induced operators F and V on $W\mathcal{O}_X = \varprojlim_R W_m\mathcal{O}_X$.

For each m , $B_m\Omega_X$ is related to $W_m\mathcal{O}_X$ via the following exact sequence, due to Serre [Se]:

$$0 \rightarrow W_m\mathcal{O}_X \xrightarrow{F} F_*W_m\mathcal{O}_X \xrightarrow{D_m} B_m\Omega_X \rightarrow 0.$$

Here D_m is defined by the formula

$$D_m: (f_0, \dots, f_{m-1}) \mapsto df_{m-1} + f_{m-2}^{p-1}df_{m-2} + \dots + f_0^{p^{m-1}-1}df_0.$$

2.2 Deformation theory in characteristic p

By a *lifting* of X to $W_2(k)$, we mean a cartesian diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad} & \tilde{X} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k) & \longrightarrow & \mathrm{Spec}(W_2(k)) \end{array}$$

where the right vertical arrow is flat and the bottom horizontal arrow is induced by the modulo p map $W_2(k) \rightarrow k$. By a *lifting of the pair* (X, F) to $W_2(k)$, we mean a pair (\tilde{X}, \tilde{F}) where \tilde{X} is a lifting of X and $\tilde{F}: \tilde{X} \rightarrow \tilde{X}$ is a morphism whose restriction to X is F and which is compatible with the Frobenius of $W_2(k)$.

It is well known that the obstruction class obs_X for the existence of a lifting of X to $W_2(k)$ lies in $\mathrm{Ext}^2(\Omega_X, \mathcal{O}_X)$. Furthermore, in Appendix of [MS], it is proved that one has the obstruction class $\mathrm{obs}_{X,F}$ in $\mathrm{Ext}^1(\Omega_X, B_1\Omega_X)$ for the existence of a lifting of the pair (X, F) to $W_2(k)$. This means that $\mathrm{obs}_{X,F}$ is zero if and only if there exists a lifting of the pair (X, F) to $W_2(k)$.

In the following, we will use the same symbol for a derivation of \mathcal{O}_X and the corresponding homomorphism from Ω_X .

Proposition 2.1. *Let \tilde{X} be a lifting of X to $W_2(k)$.*

- 1 *Let φ be an infinitesimal automorphism of \tilde{X} . Then there exists $\psi \in \mathrm{Hom}(\Omega_X, \mathcal{O}_X)$ such that $\varphi = \mathrm{id} + p\psi$.*
- 2 *Let \tilde{F}_1, \tilde{F}_2 be two liftings of the absolute Frobenius of X to \tilde{X} . Then there exists $\eta \in \mathrm{Hom}(\Omega_X, F_*\mathcal{O}_X)$ such that $\tilde{F}_1 - \tilde{F}_2 = p\eta$.*
- 3 *Let φ and ψ be as in 1. Let \tilde{F} be a lifting of the absolute Frobenius. Then $\varphi\tilde{F}\varphi^{-1}$ is also a lifting of Frobenius and we have $\tilde{F} - \varphi\tilde{F}\varphi^{-1} = p\psi^p$.*

Proof. The first statement is standard. For the others, see Proposition 1 of Appendix of [MS] and its proof. \square

For any open subscheme $U \subset X$, we denote by $Rel(X, F)(U)$ the set of isomorphism classes of liftings of the pair $(U, F|_U)$ to $W_2(k)$. Then they form a Zariski sheaf $Rel(X, F)$ on X which is a torsor under $Hom(\Omega_X, B_1\Omega_X)$ (see *ibid.*).

Similarly for any $U \subset X$, we denote by $sc(C)(U)$ the set of sections of the Cartier operator $C: Z_1\Omega_U \rightarrow \Omega_U$, i.e. \mathcal{O}_X -linear morphisms $\phi: \Omega_U \rightarrow Z_1\Omega_U$ such that the composition $C \circ \phi$ is the identity of Ω_U . Then the sheaf $sc(C)$ is also a torsor under $Hom(\Omega_X, B_1\Omega_X)$.

Proposition 2.2.

- 1 We have an isomorphism of torsors

$$Rel(X, F) \simeq sc(C).$$

- 2 The class of the extension

$$0 \rightarrow B_1\Omega_X \rightarrow Z_1\Omega_X \xrightarrow{C} \Omega_X \rightarrow 0$$

is equal to $obs_{X,F}$ in $Ext^1(\Omega_X, B_1\Omega_X)$.

- 3 Let $\delta: Ext^1(\Omega_X, B_1\Omega_X) \rightarrow Ext^2(\Omega_X, \mathcal{O}_X)$ be the connecting homomorphism induced by

$$0 \rightarrow \mathcal{O}_X \rightarrow F_*\mathcal{O}_X \xrightarrow{d} B_1\Omega_X \rightarrow 0.$$

Then $\delta(obs_{X,F}) = obs_X$.

Remark 2.3. The first assertion is a rigidified version of Theorem 3.5. of [DI]. The others are stated in [Sr] without proof.

Proof.

1. We have a morphism $Rel(X, F) \rightarrow sc(C)$ defined by the following. Let (\tilde{U}, \tilde{F}) be a lifting of $(U, F|_U)$. For a local section $x \in \mathcal{O}_{\tilde{U}}$ we have $\tilde{F}(x) = x^p + p\psi(\bar{x})$ for some function $\psi: \mathcal{O}_U \rightarrow \mathcal{O}_U$. Here $\bar{x} \in \mathcal{O}_U$ is the image of x under the reduction modulo p . Then an assignment $\bar{x} \mapsto \bar{x}^{p-1}d\bar{x} + d\psi(\bar{x})$ defines a section ϕ of the Cartier operator over U . This is a morphism of torsors under $Hom(\Omega_X, B_1\Omega_X)$, so these two are isomorphic.
2. This follows from the first assertion.
3. We first recall the construction of the obstruction classes obs_X and $obs_{X,F}$. Take an affine open covering $X = \bigcup U_i$ such that there exist liftings $(\tilde{U}_i, \tilde{F}_i)$ of the pair $(U_i, F|_{U_i})$ for each i . Let $\tilde{U}_{ij} \subset \tilde{U}_i$ be the open subscheme corresponding to $U_{ij} = U_i \cap U_j \subset U_i$. Then we have isomorphisms

$\varphi_{ij}: \tilde{U}_{ij} \xrightarrow{\sim} \tilde{U}_{ji}$ such that $\varphi_{ij}|_{U_{ij}}$ is the identity. In the following, we omit the symbol of the restriction to a smaller open subscheme. The composition $\varphi_{ik}^{-1}\varphi_{jk}\varphi_{ij}$ is an infinitesimal automorphism of \tilde{U}_{ijk} and defines a derivation $\varphi_{ijk} \in \text{Hom}(\Omega_{U_{ijk}}, \mathcal{O}_{U_{ijk}})$. Then the class $\{\varphi_{ijk}\}$ defines $\text{obs}_X \in H^2(X, \mathcal{H}om(\Omega_X, \mathcal{O}_X)) = \text{Ext}^2(\Omega_X, \mathcal{O}_X)$.

Now $\varphi_{ji}\tilde{F}_j\varphi_{ji}^{-1}$ and \tilde{F}_i are two liftings of the Frobenius on U_{ij} . So they differ by some $\eta_{ij} \in \text{Hom}(\Omega_{U_{ij}}, F_*\mathcal{O}_{U_{ij}})$ and the image $\bar{\eta}_{ij} \in \text{Hom}(\Omega_{U_{ij}}, B_1\Omega_{U_{ij}})$ of η_{ij} is independent of the choice φ_{ij} . Then $\{\bar{\eta}_{ij}\}$ defines the obstruction class $\text{obs}_{X,F} \in \text{Ext}^1(\Omega_X, B_1\Omega_X) = H^1(X, \mathcal{H}om(\Omega_X, B_1\Omega_X))$.

We have $\eta_{ij} + \eta_{jk} - \eta_{ik} = \eta_{ijk}^p$ for some $\eta_{ijk} \in \text{Hom}(\Omega_{U_{ijk}}, \mathcal{O}_{U_{ijk}})$ and the class $\{\eta_{ijk}\}$ represents $\delta(\text{obs}_{X,F})$. By definition, we have $\varphi_{ji}\tilde{F}_j\varphi_{ji}^{-1} - \tilde{F}_i = p\eta_{ij}$. From these equations, we obtain

$$\varphi_{ik}^{-1}\varphi_{jk}\varphi_{ij}\tilde{F}_i\varphi_{ij}^{-1}\varphi_{jk}^{-1}\varphi_{ik} - \tilde{F}_i = p(\eta_{ij} + \eta_{jk} - \eta_{ik}).$$

for any i, j, k . By Proposition 2.1, $\varphi_{ijk}^p = \eta_{ijk}$, so $\delta(\text{obs}_{X,F}) = \text{obs}_X$. \square

Remark 2.4. This proposition is valid for iterations of Frobenius. For each $m \geq 1$, locally the liftings of (X, F^m) have no automorphisms and form a torsor under $\mathcal{H}om(\Omega_X, F_*^m\mathcal{O}_X/\mathcal{O}_X)$. So there is the obstruction class in $\text{Ext}^1(\Omega_X, F_*^m\mathcal{O}_X/\mathcal{O}_X)$ for the existence of a lifting of (X, F^m) and obs_X is the image of this class.

2.3 Artin-Mazur height

Let n be the dimension of X . Consider the functor Φ_X from the category of Artin local k -algebras with residue field k to the category of abelian groups defined by

$$\Phi_X: A \mapsto \text{Ker}(H_{\text{ét}}^n(X \otimes_k A, \mathbb{G}_m) \rightarrow H_{\text{ét}}^n(X, \mathbb{G}_m)).$$

If X is a Calabi-Yau variety, then Φ_X is pro-represented by a one dimensional formal group ([AM]). The *Artin-Mazur height* $ht(X)$ of X is defined to be the height of the associated formal group Φ_X . The Dieudonné module of Φ_X is canonically isomorphic to the Serre's Witt vector cohomology $H^n(X, W\mathcal{O}_X)$. In particular, we see that

$$ht(X) = \begin{cases} \dim_K H^n(X, W\mathcal{O}_X) \otimes K & \text{if } H^n(X, W\mathcal{O}_X) \otimes K \neq 0, \\ \infty & \text{if } H^n(X, W\mathcal{O}_X) \otimes K = 0. \end{cases}$$

Here K is the field of fractions of $W(k)$.

This invariant has the following characterization due to T. Katsura and G. van der Geer ([GK]).

Theorem 2.5. *Let X be a Calabi-Yau variety over k of dimension n . Then we have*

$$ht(X) = \min\{m > 0; \text{the Frobenius action } F \text{ on } H^n(X, W_m\mathcal{O}_X) \text{ is non zero}\}.$$

There is another formula which will be useful for our purpose.

Proposition 2.6 ([GK] Proposition 3.1.). *Let X be a Calabi-Yau variety over k of dimension n . Then we have*

$$\dim H^i(B_m \Omega_X) = \begin{cases} 0 & \text{if } i \neq n-1, n, \\ \min\{m, ht(X) - 1\} & \text{if } i = n-1, n. \end{cases}$$

3 Definition and proof

Let X be a variety over k . For each $m > 0$, consider the following morphisms between $W_m \mathcal{O}_X$ -modules:

$$\begin{array}{ccc} W_m \mathcal{O}_X & \xrightarrow{F} & F_* W_m \mathcal{O}_X \\ R^{m-1} \downarrow & & \\ \mathcal{O}_X & & \end{array}$$

Definition 3.1. We define the *Frobenius-splitting height* $sht(X)$ of X by the minimum number $m > 0$ such that there exists a $W_m \mathcal{O}_X$ -linear homomorphism

$$\phi: F_* W_m \mathcal{O}_X \rightarrow \mathcal{O}_X$$

satisfying $\phi \circ F = R^{m-1}$. If such an m does not exist, then $sht(X)$ is ∞ . X is said to be *quasi-Frobenius-splitting* if $sht(X) < \infty$.

Remark 3.2. By the definition, X is Frobenius-splitting if and only if $sht(X) = 1$.

Remark 3.3. Assume that X is smooth over k . Consider the following exact sequence:

$$0 \rightarrow W_m \mathcal{O}_X \rightarrow F_* W_m \mathcal{O}_X \xrightarrow{D_m} B_m \Omega_X \rightarrow 0.$$

If we push out this sequence via $W_m \mathcal{O}_X \xrightarrow{R^{m-1}} \mathcal{O}_X$, then we get a new exact sequence

$$0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{F}_m \mathcal{O}_X \rightarrow B_m \Omega_X \rightarrow 0. \quad (e_m)$$

So the existence of ϕ as in the definition is equivalent to the splitting of the exact sequence (e_m) . So the Frobenius-splitting height is characterized as

$$sht(X) = \min\{m > 0; (e_m) \text{ splits}\}.$$

Furthermore, as the diagram

$$\begin{array}{ccc} W_{m+1} \mathcal{O}_X & \xrightarrow{D_{m+1}} & B_{m+1} \Omega_X \\ R \downarrow & & \downarrow C \\ W_m \mathcal{O}_X & \xrightarrow{D_m} & B_m \Omega_X \end{array}$$

commutes, the exact sequence (e_m) is obtained from the exact sequence

$$0 \rightarrow \mathcal{O}_X \rightarrow F_*\mathcal{O}_X \xrightarrow{d} B_1\Omega_X \rightarrow 0$$

by pulling back along $B_m\Omega_X \xrightarrow{C^{m-1}} B_1\Omega_X$. i.e., we have a morphism of complexes

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{F}_m\mathcal{O}_X & \longrightarrow & B_m\Omega_X \longrightarrow 0 & (e_m) \\ & & \parallel & & \downarrow & & \downarrow C^{m-1} & \\ 0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & F_*\mathcal{O}_X & \longrightarrow & B_1\Omega_X \longrightarrow 0 & (e_1). \end{array}$$

The proof of our main theorem is divided into two parts.

Theorem 3.4. *Every smooth quasi-Frobenius-splitting variety admits a flat lifting to $W_2(k)$.*

Proof. Let $\text{sht}(X) = m < \infty$. By the morphism of extensions from $(e)_m$ to $(e)_1$, we have the following commutative diagram

$$\begin{array}{ccc} \text{Ext}^1(\Omega_X, B_m\Omega_X) & \xrightarrow{\delta_m} & \text{Ext}^2(\Omega_X, \mathcal{O}_X) \\ \downarrow C_*^{m-1} & & \parallel \\ \text{Ext}^1(\Omega_X, B_1\Omega_X) & \xrightarrow{\delta} & \text{Ext}^2(\Omega_X, \mathcal{O}_X). \end{array}$$

As $(e)_m$ splits, $\delta_m = 0$. Recall that $\delta(\text{obs}_{X,F}) = \text{obs}_X$ (Proposition 2.2). So if $\text{obs}_{X,F}$ is in the image of C_*^{m-1} , the assertion holds. By the same proposition, this is valid since one has the following commutative diagram.

$$\begin{array}{ccccccc} 0 & \longrightarrow & B_m\Omega_X & \longrightarrow & Z_m\Omega_X & \xrightarrow{C^m} & \Omega_X \longrightarrow 0 \\ & & \downarrow C^{m-1} & & \downarrow C^{m-1} & & \parallel \\ 0 & \longrightarrow & B_1\Omega_X & \longrightarrow & Z_1\Omega_X & \xrightarrow{C} & \Omega_X \longrightarrow 0. \end{array}$$

□

Theorem 3.5. *For a Calabi-Yau variety X , we have*

$$\text{ht}(X) = \text{sht}(X).$$

Proof. Consider the exact sequence

$$0 \rightarrow F_*B_{m-1}\Omega_X \rightarrow B_m\Omega_X \xrightarrow{C^{m-1}} B_1\Omega_X \rightarrow 0.$$

Then one has an exact sequence

$$\begin{aligned} \text{Ext}^1(B_1\Omega_X, \mathcal{O}_X) & \xrightarrow{(C^{m-1})^*} \text{Ext}^1(B_m\Omega_X, \mathcal{O}_X) \rightarrow \text{Ext}^1(F_*B_{m-1}\Omega_X, \mathcal{O}_X) \\ & \rightarrow \text{Ext}^2(B_1\Omega_X, \mathcal{O}_X) \end{aligned}$$

and $(C^{m-1})^*((e)_1) = (e)_m$. By Serre Duality and triviality of the canonical bundle, $\text{Ext}^i(B_m\Omega_X, \mathcal{O}_X) \simeq H^{n-i}(X, B_m\Omega_X)^\vee$. Here n denotes the dimension of X . Similarly, as F is finite, $\text{Ext}^i(F_*B_{m-1}\Omega_X, \mathcal{O}_X) \simeq H^{n-i}(X, F_*B_{m-1}\Omega_X)^\vee \simeq H^{n-i}(X, B_{m-1}\Omega_X)^\vee$.

As the Frobenius-splitting is same as the Artin-Mazur height being one, using Proposition 2.6, we see that $\text{Ext}^1(B_1\Omega_X, \mathcal{O}_X)$ is generated by the class of $(e)_1$. This implies that $(e)_m = 0$ if and only if $\text{Ext}^1(B_m\Omega_X, \mathcal{O}_X) \rightarrow \text{Ext}^1(B_{m-1}\Omega_X, \mathcal{O}_X)$ is an isomorphism which is always surjective. By the same Proposition 2.6, this is same as $ht(X) \leq m$. \square

Remark 3.6. One of the most important properties of a Frobenius-splitting variety X is that, for an ample line bundle \mathcal{L} on X , we have $H^i(X, \mathcal{L}) = 0$ for $i > 0$ ([MR]). The same property holds for quasi-Frobenius-splitting varieties.

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